

RIGIDITY OF ENTIRE SELF-SHRINKING SOLUTIONS TO KÄHLER-RICCI FLOW ON COMPLEX PLANE

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ABSTRACT. We show that every entire self-shrinking solution on \mathbb{C}^1 to the Kähler-Ricci flow must be generated from a quadratic potential.

1. INTRODUCTION

In this short note, we prove the following result.

Theorem 1.1. *Suppose that $u(x)$ is an entire smooth subharmonic solution on \mathbb{R}^n to the equation*

$$(1.1) \quad \ln \Delta u = \frac{1}{2} x \cdot Du - u,$$

then u is quadratic.

For $n = 2$, up to an additive constant, equation (1.1) is equivalent to the one-dimensional case of the complex Monge-Ampère equation

$$(1.2) \quad \ln \det u_{i\bar{j}} = \frac{1}{2} x \cdot Du - u$$

on \mathbb{C}^m . Any entire solution to (1.2) leads to an entire self-shrinking solution

$$v(x, t) = -tu \left(\frac{x}{\sqrt{-t}} \right)$$

to a parabolic complex Monge-Ampère equation

$$v_t = \ln \det (v_{i\bar{j}})$$

on $\mathbb{C}^m \times (-\infty, 0)$, where $z^i = x^i + \sqrt{-1}x^{m+i}$. Note that above equation of v is the potential equation of the Kähler-Ricci flow $\partial_t g_{i\bar{j}} = -R_{i\bar{j}}$. In fact, the corresponding metric $(u_{i\bar{j}})$ is a shrinking Kähler-Ricci (non-gradient) soliton.

Assuming a certain decay of Δu —a specific completeness condition, Q. Ding and Y.L. Xin have proved Theorem (1.1) in [2]. Under the condition that the Kähler metric $(u_{i\bar{j}})$ is complete, rigidity theorem for equation (1.2) has been obtained by G. Drugan, P. Lu and Y. Yuan in [3]. Similar rigidity results for self-shrinking solutions to Lagrangian mean curvature flows in pseudo-Euclidean space were obtained in [1], [2], [4] and [5].

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Our contribution is removing extra assumptions for the rigidity of equation (1.1). As in [3] and [2], the idea of our argument is still to prove the phase $-\ln \Delta u$ is constant. Then the homogeneity of the self-similar term on the right-hand side of equation (1.1) leads to the quadratic conclusion. However, it's hard to construct a barrier function as in [3] or to find a suitable integral factor as in [2] without completeness assumption. Taking advantage of the conformality of the linearized equation (1.1), we establish a second order ordinary differential inequality for $M(r) = \max_{|x|=r} \ln \Delta u(x)$ in the sense of comparison function. Then we prove that $M(r)$ blows up in finite time by Osgood's criterion unless $\ln \Delta u$ is constant.

2. PROOF

Proof. Define the phase by

$$\phi(x) = \frac{1}{2}x \cdot Du(x) - u(x).$$

Taking two derivatives and using (1.1), we have

$$(2.1) \quad \Delta \phi = \frac{e^\phi}{2}x \cdot D\phi.$$

Define $M(r) : [0, +\infty) \rightarrow \mathbb{R}$ by

$$M(r) = \max_{|x|=r} \phi(x).$$

Assuming $\phi(x)$ is not a constant, we prove that $M(r)$ blows up in finite time.

Since M is locally Lipschitz, it is differentiable a.e. in $[0, +\infty)$. For all $r > 0$, there exists a corresponding angle $\theta_r \in \mathbb{S}^{n-1}$ satisfying

$$(2.2) \quad M(r) = \phi(r, \theta_r).$$

For $r' > 0$ small enough, we have $M(r+r') \geq \phi(r+r', \theta_r)$ and $M(r-r') \geq \phi(r-r', \theta_r)$. It follows that

$$\frac{M(r+r') - M(r)}{r'} \geq \frac{\phi(r+r', \theta_r) - \phi(r, \theta_r)}{r'}$$

and

$$\frac{M(r) - M(r-r')}{r'} \leq \frac{\phi(r, \theta_r) - \phi(r-r', \theta_r)}{r'}.$$

Letting $r' \rightarrow 0$ in above two equations, we have

$$\overline{\lim} M'_-(r) \leq \frac{\partial \phi}{\partial r}(r, \theta_r) \leq \underline{\lim} M'_+(r).$$

So if $r > 0$ is a differential point of M , we have

$$(2.3) \quad M'(r) = \frac{\partial \phi}{\partial r}(r, \theta_r).$$

Because of the maximality of $\phi(r, \theta_r)$ among $\theta \in \mathbb{S}^{n-1}$, we have $\Delta_{\mathbb{S}^{n-1}} \phi(r, \theta_r) \leq 0$. Plugin this inequality into (2.1), we obtain

$$(2.4) \quad \frac{\partial^2 \phi}{\partial r^2}(r, \theta_r) + \frac{n-1}{r} \frac{\partial \phi}{\partial r}(r, \theta_r) \geq \frac{r}{2} \exp[\phi(r, \theta_r)] \cdot \frac{\partial \phi}{\partial r}(r, \theta_r).$$

Fixing a positive R_0 , for any $r \in [0, R_0]$, $\theta \in \mathbb{S}^{n-1}$, $t \in [0, 1]$, we have the following Taylor's expansion

$$\phi(r+t, \theta) - \phi(r, \theta) \geq \frac{\partial \phi}{\partial r}(r, \theta_r) \cdot t + \frac{1}{2} \frac{\partial^2 \phi}{\partial r^2}(r, \theta_r) \cdot t^2 - Ct^3.$$

Here C is a constant depends only on R_0 , in fact we can choose $C = \max_{|x| \leq R_0+1} |D^3\phi(x)|$.

We evaluate above inequality at (r, θ_r) , where r is a differentiable point and θ_r is the corresponding critical angle. Using $M(r+t) \geq \phi(r+t, \theta_r)$, (2.2), (2.3) and (2.4), we have the following inequality that only involves M , namely

$$M(r+t) - M(r) \geq M'(r)t + \frac{1}{4} \left\{ \exp[M(r)]r - \frac{2(n-1)}{r} \right\} M'(r)t^2 - Ct^3.$$

Or equivalently,

$$(2.5) \quad \frac{1}{t^2} [M(r+t) - M(r) - M'(r)t] \geq \frac{1}{4} \left\{ \exp[M(r)]r - \frac{2(n-1)}{r} \right\} M'(r) - Ct.$$

Because inequality (2.5) holds for $r \in [0, R_0]$ a.e., we can integrate it with respect to r over any subinterval $[a, b] \subset [0, R_0]$, and get the following inequality for every $t \in [0, 1]$,

$$(2.6) \quad \begin{aligned} & \frac{1}{t^2} \left\{ \int_b^{b+t} M(r) dr - \int_a^{a+t} M(r) dr - [M(b) - M(a)]t \right\} \\ & \geq \frac{1}{4} \int_a^b \left\{ \exp[M(r)]r - \frac{2(n-1)}{r} \right\} M'(r) dr - C(b-a)t. \end{aligned}$$

Choosing differentiable points a and b and letting $t \rightarrow 0$ in (2.6), we have

$$(2.7) \quad M'(b) - M'(a) \geq \frac{1}{2} \int_a^b \left\{ \exp[M(r)]r - \frac{2(n-1)}{r} \right\} M'(r) dr.$$

Since R_0 can be arbitrarily large, in fact (2.7) holds for all differentiable points $a, b \in \mathbb{R}_+$.

We claim there exists $l_0 > 0$ such that $M'(r) > 0$ at every differentiable point in $[l_0, +\infty)$. Otherwise, there exist an increasing sequence of differentiable points $\{r_k\} \subset \mathbb{R}_+$, and a sequence of corresponding critical angles $\{\theta_k\} \subset \mathbb{S}^{n-1}$ such that

$$M'(r_k) = \frac{\partial \phi}{\partial r}(r_k, \theta_k) \leq 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} r_k = +\infty.$$

Then according to Hopf's lemma, we know $\phi(x)$ is constant in $B_{r_k}(0)$. Since r_k can be arbitrarily large, $\phi(x)$ is in fact a constant on the whole \mathbb{R}^n , which contradicts our assumption.

So there exists a certain $l_0 > 0$, such that $M'(r) > 0$ holds a.e. in $[l_0, +\infty)$. Then $M(r)$ monotonically increases on $[l_0, +\infty)$. When $a \geq l_1 \triangleq l_0 + n + 2 \exp[-M(l_0)]$, we have

$$(2.8) \quad \begin{aligned} \int_a^b \left\{ \exp[M(r)]r - \frac{2(n-1)}{r} \right\} M'(r) dr & > 2 \int_a^b \exp[M(r)] M'(r) dr \\ & = 2 \{ \exp[M(b)] - \exp[M(a)] \}. \end{aligned}$$

Combining (2.7) and (2.8), we obtain

$$(2.9) \quad M'(b) - M'(a) \geq \exp[M(b)] - \exp[M(a)].$$

Above inequality holds for all differentiable points $a, b \in [l_1, +\infty)$. Choosing a differentiable point $l_2 \geq l_1$, then $M'(r) \geq M'(l_2) > 0$ holds a.e. in $[l_2, +\infty)$. Thus $M(r) \rightarrow +\infty$ as $r \rightarrow +\infty$.

Then according to Osgood's criterion, $M(r)$ blows up in finite time, which contradicts the assumption that $\phi(x)$ is entire. Hence, we conclude $\phi(x)$ is constant. Using $\phi(x) = \frac{1}{2}x \cdot Du(x) - u(x)$, we have

$$\frac{1}{2}x \cdot D[u(x) + \phi(0)] = u(x) + \phi(0).$$

Finally, it follows from Euler's homogeneous function theorem that smooth $u(x) + \phi(0)$ is a homogeneous order 2 polynomial. \square

Remark 2.1. From the proof, it's not hard to see that the theorem also holds for

$$\Delta u = f(x \cdot Du - 2u)$$

if $f \in C^1(\mathbb{R})$ is convex, monotone increasing, and $f^{-1} \in L^1([d, +\infty))$ for a certain $d \in \mathbb{R}$. Integrability condition for f^{-1} is necessary. Otherwise, we have such counterexample: $f(x) \equiv x$ and

$$u(x) = (x_1^2 - 1) \int_0^{x_1} \frac{1}{s^2} (\exp \frac{s^2}{2} - 1) ds - \frac{1}{x_1} (\exp \frac{x_1^2}{2} - 1) - x_1.$$

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